PHYS 3090: Practice Problems for Midterm 1

These are problems similar to or the same as ones we covered during the review session in class.

Problem 1: Let $M = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Compute e^M.

Problem 2: Let $M = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$. Compute e^M. Use your result to evaluate e^M for $M = \begin{pmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{pmatrix}$.

Problem 3: Consider a diagonal matrix $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Show that $e^M = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$.

Problem 4: Let $\vec{u} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$.

- Compute $|\vec{u}|$ and $|\vec{v}|$.
- Compute $\hat{u}$ and $\hat{v}$.
- Compute $\langle iu, u\rangle$. *Answer:* $\langle iu, u\rangle = i^*\langle u, u\rangle = -2i$.
- Are $\vec{u}$ and $\vec{v}$ orthogonal? *Answer:* Yes.

Problem 5: Let $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

- Compute $R_y(\frac{3\pi}{4})\vec{v}$.
- Compute $R_z(\pi)\vec{v}$.
- Compute $R_x(\frac{\pi}{2})\vec{v}$.

We didn't cover normal modes and eigenvalue problems during the review, but this will be important. You should review all the spring problems from in class and HW assignment 2.

Here is a new problem on normal modes.

Problem 5: The following system of equations can come up in quantum mechanics when studying the time evolution of a system in a magnetic field:

$$\dot{\psi}_1 = -i\big(a\psi_1 + b\psi_2\big)\,, \quad \dot{\psi}_2 = -i\big(b\psi_1 - a\psi_2\big)\,, \tag{1}$$

where $\psi_1(t)$ and $\psi_2(t)$ are functions of time t, and a, b are constants.

- Express Eq. (1) as a matrix equation $\dot{\vec{\psi}} = -iH\vec{\psi}$, where $\vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and H is a 2×2 matrix. *Answer:* $H = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.
- Determine the eigenvalues and eigenvectors of H. *Answer:* Eigenvalues $\lambda_{1,2}$ are $\lambda_1 = \omega$ and $\lambda_2 = -\omega$, where I have defined $\omega = \sqrt{a^2 + b^2}$. The corresponding eigenvectors are $\vec{u} = \begin{pmatrix} b \\ \omega - a \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} \omega - a \\ -b \end{pmatrix}$, respectively.
- Find the normal modes of the system. *Answer:* The normal modes are

$$\vec{\psi}^{(1)}(t) = \vec{u}e^{-i\omega t}, \quad \vec{\psi}^{(2)}(t) = \vec{v}e^{i\omega t}. \tag{2}$$

  You can check this by plugging into the differential equation:

$$\dot{\vec{\psi}}^{(1)} = -iH\vec{\psi}^{(1)} = -i\omega\vec{\psi}^{(1)}, \quad \dot{\vec{\psi}}^{(2)} = -iH\vec{\psi}^{(2)} = +i\omega\vec{\psi}^{(2)}, \tag{3}$$

  since $H\vec{u} = \omega\vec{u}$ and $H\vec{v} = -\omega\vec{v}$. This problem is tricky because it's a first order differential equation, so the solutions are exponentials instead of sines and cosines.
- Find the most general solution to the system. *Answer:* It is the sum of the normal modes with arbitrary coefficients

$$\vec{\psi}(t) = A_1\vec{u}e^{-i\omega t} + A_2\vec{v}e^{i\omega t}, \tag{4}$$

  where A_1, A_2 are coeffients.